ANALYTIC DETERMINACY AND MEASURABLE CARDINALS

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1. Analytic determinacy

Definition 1 (S. Ulam). An uncountable cardinal number κ is said to be *measurable* if it carries a κ-additive $\{0,1\}$-valued diffuse measure, i.e., there is a function $\mu\colon \mathcal{P}(\kappa) \to \{0,1\}$ with the following properties.

(1) $\mu(\kappa) = 1$,
(2) $\mu(\{\eta\}) = 0$ for all $\eta \in \kappa$,
(3) $A \subseteq B \Rightarrow \mu(A) \leqslant \mu(B)$,
(4) if $\lambda < \kappa$ and $A_\xi \subseteq \kappa$ satisfy $\mu(A_\xi) = 0$ for all $\xi < \lambda$, then $\mu\Big(\bigcup_{\xi<\lambda} A_\xi\Big) = 0$.

We shall not enter into the study of measurable cardinals, but will only need one property of these. For this we shall need the following notation: If X is a set and m a natural number, we let $[X]^m$ denote the collection of m-element subsets of X.

Theorem 2 (F. Rowbottom). *Suppose κ is a measurable cardinal. Then for any family $\{f_i\}_{i\in\mathbb{N}}$ of functions $f_i\colon [\kappa]^{m_i} \to \mathbb{N}$, there is a subset $X \subseteq \kappa$ of cardinality $\aleph_1$ that is* monochromatic *for all f_i, i.e., each f_i is constant on $[X]^{m_i}$.*

It is a consequence of K. Gödel's second incompleteness theorem that the existence of measurable cardinals cannot be shown in ZFC. This follows from the fact that using a measurable cardinal κ, one can construct a *set model* V_κ of ZFC, which would then imply that ZFC is consistent. On the other hand, despite 80 years of research, no one has thus far been able to prove that measurable cardinals do not exist and the structural knowledge surrounding these cardinals at least give some indication that no such contradiction will ever be found.

Theorem 3 (D. A. Martin). *Assume that there exists a measurable cardinal κ. Then analytic games are determined.*

Proof. Recall that the Kleene-Brouwer ordering $<_{KB}$ is a strict linear order on $\mathbb{N}^{<\mathbb{N}}$ with the property that if $S \subseteq \mathbb{N}^{<\mathbb{N}}$ is a tree, then S is wellfounded if and only if S is wellordered under $<_{KB}$, which happens if and only if $(S, <_{KB})$ embeds into $(\omega_1, <)$. Fix also an enumeration $u_0, u_1, u_2, \ldots$ of $\mathbb{N}^{<\mathbb{N}}$ such that $|u_n| \leqslant n$.

Assume now that A is an analytic subset of $\mathbb{N}^{\mathbb{N}}$ and let T be a tree on $\mathbb{N}\times\mathbb{N}$ such that $x \in A \Leftrightarrow \exists y \in \mathbb{N}^{\mathbb{N}}\ (x,y) \in [T]$, i.e.,

$$x \notin A \Leftrightarrow (T(x), <_{KB}) \text{ embeds into } (\omega_1, <),$$

where $T(x) = \{u \in \mathbb{N}^{<\mathbb{N}} \mid (x|_{|u|}, u) \in T\}$ is the section tree. We shall show that the game $G(A)$ below, in which **I** and **II** alternate in playing $x_i \in \mathbb{N}$, is determined.

$$\begin{array}{lcccccccc} \textbf{I} & x_0 & & x_2 & & x_4 & & \cdots \\ \textbf{II} & & x_1 & & x_3 & & x_5 & \cdots \end{array}$$

Here **II** wins a run of the game if $x = (x_0, x_1, x_2, \ldots) \notin A$.

To do this, we introduce another game $G^\star(A)$ in which **I** and **II** alternate in playing $x_{2i} \in \mathbb{N}$, respectively pairs $(x_{2i+1}, \eta_i) \in \mathbb{N} \times \kappa$,

$$\begin{array}{lccccccc} \mathbf{I} & x_0 & & x_2 & & x_4 & & \cdots \\ \mathbf{II} & & x_1, \eta_0 & & x_3, \eta_1 & & x_5, \eta_2 & \cdots \end{array}$$

Again, **II** wins a run of the game $G^\star(A)$ if the following holds

$$\begin{aligned} \eta_i = 0 \quad & \text{for } u_i \notin T(x), \\ u_i <_{KB} u_j \iff \eta_i < \eta_j \quad & \text{for all } u_i, u_j \in T(x). \end{aligned}$$

Therefore, if **II** wins a run of $G^\star(A)$, then not only is $T(x)$ wellfounded and hence $x \notin A$, but **II** has simultaneously produced a witness to this effect, that is, an embedding $u_i \mapsto \eta_i$ of $(T(x), <_{KB})$ into $(\kappa, <)$. Thus, if **II** has a winning strategy in $G^\star(A)$, he also has a winning strategy in $G(A)$.

Note now that, as opposed to $G(A)$, the game $G^\star(A)$ is open for **I** and hence determined. It therefore suffices to show that if **I** has a winning strategy $\sigma^\star$ in $G^\star(A)$, he also has a winning strategy σ in $G(A)$. So assume that such a strategy $\sigma^\star$ is given.

For any $s \in \mathbb{N}^{2n}$, we define $D_s = \{u_i \mid i < n \;\&\; (s|_{|u_i|}, u_i) \in T\}$ and note that if $s \subseteq t$, then $D_s \subseteq D_t$. Moreover, for any $x \in \mathbb{N}^{\mathbb{N}}$, we have $u_i \in T(x) \iff u_i \in \bigcup_{s \subset x} D_s$.

Assume now that $s \in \mathbb{N}^{2n}$ with $|D_s| = m$ is given. Then for any $Q \in [\kappa]^m$, there is a **unique** function mapping $i < n$ to some $\xi_i^{s,Q} < \kappa$ such that

$$\begin{aligned} \xi_i^{s,Q} = 0 \quad & \text{for } u_i \notin D_s, \\ \xi_i^{s,Q} \in Q \quad & \text{for } u_i \in D_s, \end{aligned}$$

and

$$u_i <_{KB} u_j \iff \xi_i^{s,Q} < \xi_j^{s,Q} \quad \text{for all } u_i, u_j \in D_s.$$

We can therefore define a colouring $f_s \colon [\kappa]^m \to \mathbb{N}$ by

$$f_s(Q) = \sigma^\star\Big(s_0, s_1, \xi_0^{s,Q}, s_2, s_3, \xi_1^{s,Q}, \ldots, s_{2n-2}, s_{2n-1}, \xi_{n-1}^{s,Q}\Big).$$

By Rowbottom's Theorem, we can find a subset $X \subseteq \kappa$ of cardinality $\aleph_1$ that is monochromatic for all colourings f_s with $s \in \mathbb{N}^{<\mathbb{N}}$ of even length. It follows that we can unambiguously define a strategy σ for **I** in $G(A)$ as follows: If $s \in \mathbb{N}^{2n}$ with $|D_s| = m$,

$$\sigma\Big(s_0, s_1, \ldots, s_{2n-2}, s_{2n-1}\Big) = f_s(Q) = \sigma^\star\Big(s_0, s_1, \xi_0^{s,Q}, \ldots, s_{2n-2}, s_{2n-1}, \xi_{n-1}^{s,Q}\Big)$$

for **all** choices of $Q \in [X]^m$.

We claim that σ is a strategy for **I** to play in A. To see this, suppose towards a contradiction that $x \notin A$ is a run of $G(A)$ in which **I** has followed σ. Then $(T(x), <_{KB})$ embeds into $(\omega_1, <)$ and since $X \subseteq \kappa$ has cardinality $\aleph_1$, we can find an embedding $\eta \colon (T(x), <_{KB}) \to (X, <)$ and extend η to all of $\mathbb{N}^{<\mathbb{N}}$ by setting $\eta(t) = 0$ for $t \notin T(x)$. A moment of reflection is now enough to see that

$$\begin{array}{lccccccc} \mathbf{I} & x_0 & & x_2 & & x_4 & & \cdots \\ \mathbf{II} & & x_1, \eta(u_0) & & x_3, \eta(u_1) & & x_5, \eta(u_2) & \cdots \end{array}$$

is played according to the strategy $\sigma^\star$ and hence $x \in A$, which is absurd. $\square$

2. Turing reducibility and AD

A function $\phi\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ is said to be *monotone* if $s \subseteq t \Rightarrow \phi(s) \subseteq \phi(t)$. Note that in this case, the set

$$D_\phi = \{x \in \mathbb{N}^\mathbb{N} \mid |\phi(x|_n)| \underset{n\to\infty}{\longrightarrow} \infty\}$$

is G_δ and ϕ induces a continuous function $\phi^\star\colon D_\phi \to \mathbb{N}^\mathbb{N}$ by setting $\phi^\star(x) = \bigcup_n \phi(x|_n)$. We say that ϕ is *computable* or *recursive* if it is given by an algorithm, i.e., if there is a computer program that on input s outputs $\phi(s)$. Since there are only countable many computer programs, there are only countably many computable $\phi\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$, which we can list as $\{\phi_n\}$.

Using this, we can define an Σ^0_3 quasiordering $\leqslant_T$ of $\mathbb{N}^\mathbb{N}$ as follows.

$$x \leqslant_T y \Leftrightarrow \exists \phi \text{ computable } \quad \phi^\star(y) = x.$$

Thus, $x \leqslant_T y$ holds exactly when there is an algorithm computing x from y. Since the identity function on $\mathbb{N}^{<\mathbb{N}}$ is computable and the class of computable functions is closed under composition, $\leqslant_T$ is indeed both reflexive and transitive. Moreover, since there are only countably many computable functions, for any $y \in \mathbb{N}^\mathbb{N}$, the initial segment

$$I_y = \{x \in \mathbb{N}^\mathbb{N} \mid x \leqslant_T y\}$$

is countable and so $\forall y\ \forall^* x\ \ x \not\leqslant_T y$. Therefore, using the Kuratowski–Ulam theorem, we have $\forall^* x\ \forall^* y\ \ x \not\leqslant_T y$, meaning that for all but a meagre set of $x \in \mathbb{N}^\mathbb{N}$, the *cone* $C_x = \{y \in \mathbb{N}^\mathbb{N} \mid x \leqslant_T y\}$ is meagre. In particular, picking any $z \notin I_x \cup C_x$, we find that x and z are incomparable with respect to $\leqslant_T$. So $\leqslant_T$ is not total.

Any sequence $x_0, x_1, \ldots \in \mathbb{N}^\mathbb{N}$ has an upper bound in $\leqslant_T$. For if $\langle\cdot,\cdot\rangle\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ is a computable bijection, we can define $y \in \mathbb{N}^\mathbb{N}$ by $y(\langle n,m\rangle) = x_n(m)$ and see that all of the x_n are computable from y, i.e., $x_n \leqslant_T y$. Using this, it follows that $C_y \subseteq \bigcap_n C_{x_n}$.

The quasiordering $\leqslant_T$ induces the *Turing equivalence relation* $\equiv_T$ by

$$x \equiv_T y \Leftrightarrow x \leqslant_T y \ \&\ y \leqslant_T x.$$

So x and y are Turing equivalent if they can be computed from each other and thus can be understood as having the same informational content. Note that $\equiv_T$ is a $\boldsymbol{\Sigma}^0_3$ equivalence relation with countable classes.

Theorem 4 (D. A. Martin)**.** *Suppose $A \subseteq \mathbb{N}^\mathbb{N}$ is a $\equiv_T$-invariant Borel set. Then A either contains a cone or is disjoint from a cone.*

Proof. Consider the usual game $G(A)$ for playing in A,

$$\begin{array}{lccccccc} \mathbf{I} & x_0 & & x_1 & & x_2 & & \cdots \\ \mathbf{II} & & y_0 & & y_1 & & y_2 & \cdots \end{array}$$

with outcome $(x_0, y_0, x_1, y_1, \ldots)$.

Assume first that **I** has a strategy $\sigma\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ for playing in A. Fixing a computable bijection $\tau\colon \mathbb{N} \to \mathbb{N}^{<\mathbb{N}}$, we see that $z = \sigma\circ\tau \in \mathbb{N}^\mathbb{N}$ and claim that $C_z \subseteq A$. To see this, suppose that $z \leqslant_T y = (y_0, y_1, \ldots)$ and let **II** play the sequence $y = (y_0, y_1, \ldots)$ in $G(A)$, while **I** responds with $x = (x_0, x_1, \ldots)$ according to the strategy σ. Now since τ is a computable bijection, $\sigma = z \circ \tau^{-1}$ is computable from z, whereby x is computable from the pair (z, y). So, since $z \leqslant_T y$, it follows that $x \leqslant_T y$ and $(x_0, y_0, x_1, y_1, \ldots) \equiv_T y$. Moreover, since σ is winning for **I**, $(x_0, y_0, x_1, y_1, \ldots) \in A$, whence by $\equiv_T$-invariance, $y \in A$. So $C_z \subseteq A$ and A contains a cone. A similar argument shows that if **II** has a winning strategy, then $C_z \cap A = \emptyset$ for some $z \in \mathbb{N}^{\mathbb{N}}$. Finally, since A is Borel the game is determined. □

Corollary 5. *Any $\equiv_T$-invariant Borel map $f\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{R}$ is constant on a cone.*

Proof. Let (U_n) be a basis for the topology on $\mathbb{R}$ and find inductively $x_0 \leqslant_T x_1 \leqslant_T \ldots$ such that for all n¡ either $C_{x_n} \subseteq f^{-1}(U_n)$ or $C_{x_n} \cap f^{-1}(U_n) = \emptyset$. Picking y such that $x_n \leqslant_T y$ for all n, we see that f must be constant on C_y. □

Note that the proofs above only rely on the fact that $G(A)$ is determined. So, as a corollary, we see that under AD any $\equiv_T$-invariant set either contains or is disjoint from a cone. We shall now use this to show that AD implies that $\aleph_1$ is a measurable cardinal.

Theorem 6 (R. M. Solovay)**.** *If AD holds, then $\aleph_1$ is a measurable cardinal.*

Proof. Using any bijection between $\mathbb{Q}$ and $\mathbb{N}$, we can identify $\mathbb{N}^{\mathbb{Q}}$ with $\mathbb{N}^{\mathbb{N}}$ and similarly transfer $\equiv_T$ to $\mathbb{N}^{\mathbb{Q}}$. We also recall that WO is the collection of wellordered subsets of $\mathbb{Q}$ identified with a subset of $2^{\mathbb{Q}} \subseteq \mathbb{N}^{\mathbb{Q}}$. For every $y \in WO$, we let $\mathrm{otp}(y)$ denote its ordertype, which is a countable ordinal.

Define a cofinal map $\vartheta\colon \mathbb{N}^{\mathbb{Q}} \to \omega_1$ by $\vartheta(x) = \sup\{\mathrm{otp}(y) \mid y \in WO \ \&\ y \leqslant_T x\}$. So clearly $x \leqslant_T z \implies \vartheta(x) \leqslant \vartheta(z)$, i.e., ϑ is monotone.

We now define a countably additive $\{0,1\}$-valued diffuse measure μ on $\aleph_1 = \omega_1$ by letting

$$\mu(B) = \begin{cases} 1 & \text{if } \vartheta^{-1}(B) \text{ contains a cone,} \\ 0 & \text{if } \vartheta^{-1}(B) \text{ is disjoint from a cone.} \end{cases}$$

Monotonicity of μ is clear. Also, to see that μ is diffuse, suppose towards a contradiction that $\mu(\{\xi\}) = 1$ for some $\xi < \omega_1$ and find a cone C_x contained in $\vartheta^{-1}(\{\xi\})$. Now pick any $y \in WO$ with $\mathrm{otp}(y) > \xi$ and choose z such that $x, y \leqslant_T z$. Then $z \in C_x$, but $\vartheta(z) > \xi$, contradicting our assumption. Finally, μ is countably additive since the intersection of countably many cones again contains a cone. □